

Exercise 3

Derive the integration formula obtained in Exercise 2 by integrating the branch

$$f(z) = \frac{z^{-1/2}}{z^2 + 1} = \frac{e^{(-1/2)\log z}}{z^2 + 1} \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the multiple-valued function $z^{-1/2}/(z^2 + 1)$ over the closed contour in Fig. 110 (Sec. 91).

Solution

In order to evaluate this integral, consider the given function in the complex plane $f(z)$ and the contour in Fig. 110. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} z^2 + 1 &= 0 \\ z &= \pm i \end{aligned}$$

Since $z^{-1/2}$ can be written in terms of $\log z$, a branch cut for the function needs to be chosen.

$$z^{-1/2} = \exp\left(-\frac{1}{2}\log z\right)$$

We choose it to be the axis of positive real numbers.

$$= \exp\left[-\frac{1}{2}(\ln r + i\theta)\right], \quad (|z| > 0, 0 < \theta < 2\pi),$$

where $r = |z|$ is the magnitude of z and $\theta = \arg z$ is the argument of z .

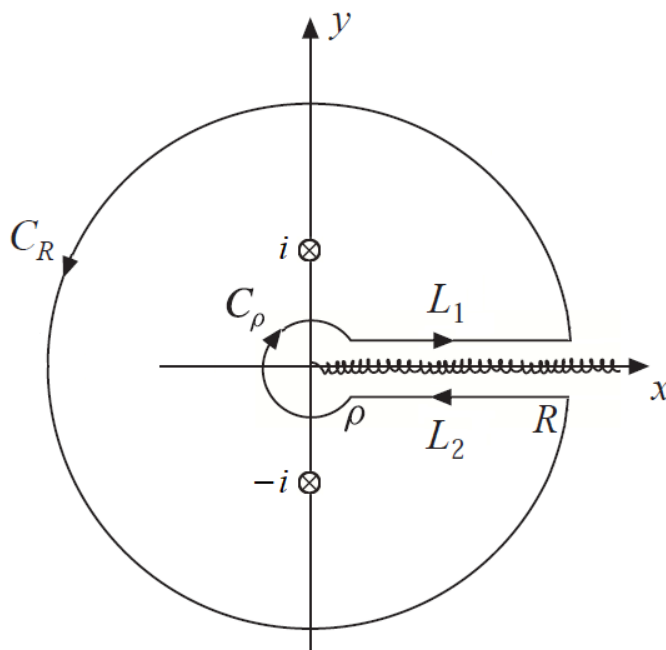


Figure 1: This is essentially Fig. 110 with the singularities at $z = -i$ and $z = i$ marked. The squiggly line represents the branch cut ($|z| > 0, 0 < \theta < 2\pi$).

According to Cauchy's residue theorem, the integral of $z^{-1/2}/(z^2 + 1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{-1/2}}{z^2 + 1} dz = 2\pi i \left(\operatorname{Res}_{z=-i} \frac{z^{-1/2}}{z^2 + 1} + \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} \right)$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\begin{aligned} \int_{L_1} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{L_2} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz \\ = 2\pi i \left(\operatorname{Res}_{z=-i} \frac{z^{-1/2}}{z^2 + 1} + \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} \right) \quad (1) \end{aligned}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1 : \quad z &= r e^{i0}, & r = \rho &\rightarrow r = R \\ L_2 : \quad z &= r e^{i2\pi}, & r = R &\rightarrow r = \rho \\ C_\rho : \quad z &= \rho e^{i\theta}, & \theta = 2\pi &\rightarrow \theta = 0 \\ C_R : \quad z &= R e^{i\theta}, & \theta = 0 &\rightarrow \theta = 2\pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{L_1} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{L_2} \frac{z^{-1/2}}{z^2 + 1} dz &= \int_\rho^R \frac{(r e^{i0})^{-1/2}}{(r e^{i0})^2 + 1} (dr e^{i0}) + \int_R^\rho \frac{(r e^{i2\pi})^{-1/2}}{(r e^{i2\pi})^2 + 1} (dr e^{i2\pi}) \\ &= \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr + \int_R^\rho \frac{r^{-1/2} e^{-i\pi}}{r^2 + 1} dr \\ &= \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr + \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr \\ &= 2 \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr. \end{aligned}$$

Substitute this formula into equation (1).

$$2 \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz = 2\pi i \left(\operatorname{Res}_{z=-i} \frac{z^{-1/2}}{z^2 + 1} + \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} \right)$$

Take the limit now as $\rho \rightarrow 0$ and $R \rightarrow \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$2 \int_0^\infty \frac{r^{-1/2}}{r^2 + 1} dr = 2\pi i \left(\operatorname{Res}_{z=-i} \frac{z^{-1/2}}{z^2 + 1} + \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} \right)$$

Writing the denominator as $z^2 + 1 = (z + i)(z - i)$, we see that the multiplicities of the $z - i$ and $z + i$ factors are both 1. Consequently, the residues at $z = -i$ and $z = i$ can be calculated by

$$\begin{aligned} \operatorname{Res}_{z=-i} \frac{z^{-1/2}}{z^2 + 1} &= \phi_1(-i) \\ \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} &= \phi_2(i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are the same function as $f(z)$ without the $z+i$ and $z-i$ factors, respectively.

$$\begin{aligned}\phi_1(z) = \frac{z^{-1/2}}{z-i} &\Rightarrow \phi_1(-i) = \frac{(-i)^{-1/2}}{-2i} = \frac{(e^{i3\pi/2})^{-1/2}}{-2i} = -\frac{e^{-i3\pi/4}}{2i} \\ \phi_2(z) = \frac{z^{-1/2}}{z+i} &\Rightarrow \phi_2(i) = \frac{i^{-1/2}}{2i} = \frac{(e^{i\pi/2})^{-1/2}}{2i} = \frac{e^{-i\pi/4}}{2i}\end{aligned}$$

So then

$$\begin{aligned}\operatorname{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1} &= -\frac{e^{-i3\pi/4}}{2i} \\ \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} &= \frac{e^{-i\pi/4}}{2i}\end{aligned}$$

and

$$\begin{aligned}2 \int_0^\infty \frac{r^{-1/2}}{r^2+1} dr &= 2\pi i \left(-\frac{e^{-i3\pi/4}}{2i} + \frac{e^{-i\pi/4}}{2i} \right) \\ &= \pi(e^{-i\pi/4} - e^{-i3\pi/4}) \\ &= \pi \left[\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} - \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right) \right] \\ &= \pi \left[\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) \right] \\ &= \pi\sqrt{2}.\end{aligned}$$

Divide both sides by 2.

$$\int_0^\infty \frac{r^{-1/2}}{r^2+1} dr = \frac{\pi}{\sqrt{2}}$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)} dx = \frac{\pi}{\sqrt{2}}.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \rightarrow 0$. The parameterization of the small circular arc in Figure 1 is $z = \rho e^{i\theta}$, where θ goes from 2π to 0 .

$$\begin{aligned}\int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz &= \int_{2\pi}^0 \frac{(\rho e^{i\theta})^{-1/2}}{(\rho e^{i\theta})^2 + 1} (\rho i e^{i\theta} d\theta) \\ &= \int_{2\pi}^0 \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta} + 1} (i d\theta)\end{aligned}$$

Take the limit of both sides as $\rho \rightarrow 0$.

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz = \lim_{\rho \rightarrow 0} \int_{2\pi}^0 \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta} + 1} (i d\theta)$$

The limits of integration are constant, so the limit may be brought inside the integral.

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz = \int_{2\pi}^0 \lim_{\rho \rightarrow 0} \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta} + 1} (i d\theta)$$

Because of $\rho^{1/2}$ in the numerator, the limit is zero. Therefore,

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the large circular arc in Figure 1 is $z = R e^{i\theta}$, where θ goes from 0 to 2π .

$$\begin{aligned}\int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz &= \int_0^{2\pi} \frac{(R e^{i\theta})^{-1/2}}{(R e^{i\theta})^2 + 1} (R i e^{i\theta} d\theta) \\ &= \int_0^{2\pi} \frac{R^{1/2} e^{i\theta/2}}{R^2 e^{2i\theta} + 1} (i d\theta) \\ &= \int_0^{2\pi} \frac{R^{1/2} e^{i\theta/2}}{R^2 (e^{2i\theta} + \frac{1}{R^2})} (i d\theta) \\ &= \int_0^{2\pi} \frac{1}{R^{3/2}} \frac{e^{i\theta/2}}{(e^{2i\theta} + \frac{1}{R^2})} (i d\theta)\end{aligned}$$

Take the limit of both sides as $R \rightarrow \infty$. Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz = \int_0^{2\pi} \lim_{R \rightarrow \infty} \frac{1}{R^{3/2}} \frac{e^{i\theta/2}}{(e^{2i\theta} + \frac{1}{R^2})} (i d\theta)$$

Because of $R^{3/2}$ in the denominator, the limit is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz = 0.$$